Computational Statistics and Data Analysis 58 (2013) 352–367

Contents lists available at SciVerse ScienceDirect

Computational Statistics and Data Analysis

journal homepage: www.elsevier.com/locate/csda

The compound class of extended Weibull power series distributions

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ARTICLE INFO

Article history:
Received 2 March 2012
Received in revised form 7 September 2012
Accepted 14 September 2012
Available online 25 September 2012

Keywords:
Extended Weibull distribution
Extended Weibull power series distribution
Order statistic
Power series distribution

ABSTRACT

We introduce a general method for obtaining more flexible new distributions by compounding the extended Weibull and power series distributions. The compounding procedure follows the same set-up carried out by Adamidis and Loukas (1998) and defines 68 new sub-models. The new class of generated distributions includes some well-known mixing distributions, such as the Weibull power series (Morais and Barreto-Souza, 2011) and exponential power series (Chahkandi and Ganjali, 2009) distributions. Some mathematical properties of the new class are studied including moments and the generating function. We provide the density function of the order statistics and their moments. The method of maximum likelihood is used for estimating the model parameters. Special distributions are investigated. We illustrate the usefulness of the new distributions by means of two applications to real data sets.

1. Introduction

The modeling and analysis of lifetimes is an important aspect of statistical work in a wide variety of scientific and technological fields. Several distributions have been proposed in the literature to model lifetime data by compounding some useful lifetime distributions. Adamidis and Loukas (1998) introduced a two-parameter exponential geometric (EG) distribution by compounding the exponential and geometric distributions. In a similar manner, the exponential Poisson (EP) and exponential logarithmic (EL) distributions were introduced and studied by Kus (2007) and Tahmasbi and Rezaei (2008), respectively. Recently, Chahkandi and Ganjali (2009) have proposed the exponential power series (EPS) family of distributions, which contains as special cases these distributions. Barreto-Souza et al. (2010) and Lu and Shi (2011) introduced the Weibull geometric (WG) and Weibull Poisson (WP) distributions which naturally extend the EG and EP distributions, respectively. In a very recent paper, Morais and Barreto-Souza (2011) defined the Weibull power series (WPS) class of distributions which includes as sub-models the EPS distributions. The WPS distributions can have an increasing, decreasing and upside down bathtub failure rate function. The generalized exponential power series (GEPS) distributions were proposed by Mahmoudi and Jafari (2012) following the same approach developed by Morais and Barreto-Souza (2011). Another recent compounded distribution can be found in Cancho et al. (2011, 2012) that introduced the Poisson exponential (PE) and geometric Birnbaum–Saunders (GBS) distributions and Barreto-Souza and Bakouch (2012) who defined the Poisson Lindley (PL) distribution. Further, Louzada et al. (2011) and Cordeiro et al. (2012) proposed the complementary exponential geometric (CEG) and the exponential COM Poisson (ECOMP) distributions.

The Weibull distribution was one of the earliest and most popular model for failure times. In recent years, many authors have proposed generalizations of the Weibull model based on extended types of failure of a system. In the context of modeling random strength of brittle materials and failure times, Gurvich et al. (1997) proposed an extended Weibull (EW)

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0167-9473/$ – see front matter
doi:10.1016/j.csda.2012.09.009

family of distributions. Nadarajah and Kotz (2005) and Pham and Lai (2007) presented much more than twenty useful distributions in their family. Its cumulative distribution function (cdf) is given by

$$G(x;\ \alpha,\boldsymbol{\xi}) = 1 - \mathrm{e}^{-\alpha\, H(x;\, \boldsymbol{\xi})}, \quad x > 0,\ \alpha > 0, \tag{1}$$

where $H(x;\ \boldsymbol{\xi})$ is a non-negative monotonically increasing function which depends on a parameter vector $\boldsymbol{\xi}$. The corresponding probability density function (pdf) becomes

$$g(x;\ \alpha,\boldsymbol{\xi}) = \alpha\, h(x;\ \boldsymbol{\xi})\, \mathrm{e}^{-\alpha\, H(x;\, \boldsymbol{\xi})}, \quad x > 0,\ \alpha > 0, \tag{2}$$

where $h(x;\ \boldsymbol{\xi})$ is the first derivative of $H(x;\ \boldsymbol{\xi})$. Many well-known models are special cases of Eq. (1) such as:

(i) $H(x;\boldsymbol{\xi}) = x$ gives the exponential distribution;
(ii) $H(x;\boldsymbol{\xi}) = x^2$ yields the Rayleigh distribution (Burr type-X distribution);
(iii) $H(x;\boldsymbol{\xi}) = \log(x/k)$ leads to the Pareto distribution;
(iv) $H(x;\boldsymbol{\xi}) = \beta^{-1}[\exp(\beta x) - 1]$ gives the Gompertz distribution.

We emphasize that several other distributions could be re-written in form (1) (see some examples in Nadarajah and Kotz, 2005; Pham and Lai, 2007). In this paper, we define the extended Weibull power series (EWPS) class of univariate distributions obtained by compounding the extended Weibull and power series distributions. The compounding procedure follows the key idea of Adamidis and Loukas (1998) or, more generally, by Chahkandi and Ganjali (2009) and Morais and Barreto-Souza (2011). The new class of distributions includes as special models the WPS distributions, which in turn extends the EPS distributions and defines 68 (17 × 4) new sub-models as special cases. The hazard function of the proposed class can be decreasing, increasing, bathtub and upside down bathtub. We are motivated to introduce the EWPS distributions because of the wide usage of (1) and the fact that the current generalization provides means of its continuous extension to still more complex situations.

This paper is organized as follows. In Section 2, we define the EWPS class of distributions and demonstrate that there are many existing models which can be deduced as special cases of the proposed unified model. In Section 3, we provide general properties of the EWPS class including the density, survival and hazard rate functions, some useful expansions, quantiles, ordinary and incomplete moments, generating function, order statistics and their moments, reliability and average lifetime. Estimation of the parameters by maximum likelihood is investigated in Section 4. In Section 5, we present suitable constraints leading to the maximum entropy characterization of the new class. Three special models of the proposed class are studied in Section 6. Applications to two real data sets are presented in Section 7. Some concluding remarks are addressed in Section 8.

2. The new class

Let N be a discrete random variable having a power series distribution (truncated at zero) with probability mass function

$$p_n = P(N = n) = \frac{a_n\, \theta^n}{C(\theta)}, \quad n = 1, 2, \ldots, \tag{3}$$

where a_n depends only on n, $C(\theta) = \sum_{n=1}^{\infty} a_n\, \theta^n$ and $\theta > 0$ is such that $C(\theta)$ is finite. The proposed class of distributions can be derived as follows. Given N, let $X_1, \ldots, X_N$ be independent and identically distributed (iid) random variables following (1). Table 1 summarizes some power series distributions (truncated at zero) defined according to (3) such as the Poisson, logarithmic, geometric and binomial distributions. Let $X_{(1)} = \min\{X_i\}_{i=1}^{N}$. The conditional cumulative distribution of $X_{(1)}|N = n$ is given by

$$G_{X_{(1)}|N=n}(x) = 1 - \mathrm{e}^{-n\alpha H(x;\boldsymbol{\xi})},$$

i.e., $X_{(1)}|N = n$ has the general class of distributions (1) with parameters $n\alpha$ and $\boldsymbol{\xi}$ based on the same $H(x;\ \boldsymbol{\xi})$ function. Hence, we obtain

$$P(X_{(1)} \le x, N = n) = \frac{a_n\, \theta^n}{C(\theta)} \left[1 - \mathrm{e}^{-n\alpha H(x;\boldsymbol{\xi})}\right], \quad x > 0,\ n \ge 1.$$

The EWPS class of distributions is defined by the marginal cdf of $X_{(1)}$:

$$F(x;\theta,\alpha,\boldsymbol{\xi}) = 1 - \frac{C(\theta\, \mathrm{e}^{-\alpha H(x;\boldsymbol{\xi})})}{C(\theta)}, \quad x > 0. \tag{4}$$

We provide at least four motivations for the EWPS class of distributions, which can be applied in some interesting situations as follows:

1. Time to the first failure. Suppose that the failure of a device occurs due to the presence of an unknown number N of initial defects of same kind, which can be identifiable only after causing failure and are repaired perfectly. Define by X_i

Table 1
Useful quantities for some power series distributions.

Distribution	a_n	$C(\theta)$	$C'(\theta)$	$C''(\theta)$	$C(\theta)^{-1}$	Θ
Poisson	$n!^{-1}$	$e^{\theta}-1$	e^{θ}	e^{θ}	$\log(\theta+1)$	$\theta\in(0,\infty)$
Logarithmic	n^{-1}	$-\log(1-\theta)$	$(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$1-e^{-\theta}$	$\theta\in(0,1)$
Geometric	1	$\theta(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$2(1-\theta)^{-3}$	$\theta(\theta+1)^{-1}$	$\theta\in(0,1)$
Binomial	$\binom{m}{n}$	$(\theta+1)^m-1$	$m(\theta+1)^{m-1}$	$\frac{m(m-1)}{(\theta+1)^{2-m}}$	$(\theta-1)^{1/m}-1$	$\theta\in(0,1)$

Table 2
Special distributions and corresponding $H(x;\ \boldsymbol{\xi})$ and $h(x;\ \boldsymbol{\xi})$ functions.

Distribution	$H(x;\boldsymbol{\xi})$	$h(x;\boldsymbol{\xi})$	α	$\boldsymbol{\xi}$	References
Exponential ($x\geq 0$)	x	1	α	$\emptyset$	Johnson et al. (1994)
Pareto ($x\geq k$)	$\log(x/k)$	$1/x$	α	k	Johnson et al. (1994)
Rayleigh ($x\geq 0$)	x^2	$2x$	α	$\emptyset$	Rayleigh (1880)
Weibull ($x\geq 0$)	x^{γ}	$\gamma x^{\gamma-1}$	α	γ	Johnson et al. (1994)
Modified Weibull ($x\geq 0$)	$x^{\gamma}\exp(\lambda x)$	$x^{\gamma-1}\exp(\lambda x)(\gamma+\lambda x)$	α	$[\gamma,\lambda]$	Lai et al. (2003)
Weibull extension ($x\geq 0$)	$\lambda[\exp(x/\lambda)^{\beta}-1]$	$\beta\exp(x/\lambda)^{\beta}(x/\lambda)^{\beta-1}$	α	$[\gamma,\lambda,\beta]$	Xie et al. (2002)
Log–Weibull ($-\infty<x<\infty$)	$\exp[(x-\mu)/\sigma]$	$(1/\sigma)\exp[(x-\mu)/\sigma]$	1	$[\mu,\sigma]$	White (1969)
Phani ($0<\mu<x<\sigma<\infty$)	$[(x-\mu)/(\sigma-x)]^{\beta}$	$\beta[(x-\mu)/(\sigma-x)]^{\beta-1}[(\sigma-\mu)/(\sigma-t)^2]$	α	$[\mu,\sigma,\beta]$	Phani (1987)
Weibull Kies ($0<\mu<x<\sigma<\infty$)	$(x-\mu)^{\beta_1}/(\sigma-x)^{\beta_2}$	$(x-\mu)^{\beta_1-1}(\sigma-x)^{-\beta_2-1}[\beta_1(\sigma-x)+\beta_2(x-\mu)]$	α	$[\mu,\sigma,\beta_1,\beta_2]$	Kies (1958)
Additive Weibull ($x\geq 0$)	$(x/\beta_1)^{\alpha_1}+(x/\beta_2)^{\alpha_2}$	$(\alpha_1/\beta_1)(x/\beta_1)^{\alpha_1-1}+(\alpha_2/\beta_2)(x/\beta_2)^{\alpha_2-1}$	1	$[\alpha_1,\alpha_2,\beta_1,\beta_2]$	Xie and Lai (1995)
Traditional Weibull ($x\geq 0$)	$x^b[\exp(cx^d-1)]$	$bx^{b-1}[\exp(cx^d)-1]+cdx^{b+d-1}\exp(cx^d)$	α	$[b,c,d]$	Nadarajah and Kotz (2005)
Gen. power Weibull ($x\geq 0$)	$[1+(x/\beta)^{\alpha_1}]^{\theta}-1$	$(\theta\alpha_1/\beta)[1+(x/\beta)^{\alpha_1}]^{\theta-1}(x/\beta)^{\alpha_1}$	1	$[\alpha_1,\beta,\theta]$	Nikulin and Haghighi (2006)
Flexible Weibull extension ($x\geq 0$)	$\exp(\alpha_1 x-\beta/x)$	$\exp(\alpha_1 x-\beta/x)(\alpha_1+\beta/x^2)$	1	$[\alpha_1,\beta]$	Bebbington et al. (2007)
Gompertz ($x\geq 0$)	$\beta^{-1}[\exp(\beta x)-1]$	$\exp(\beta x)$	α	β	Gompertz (1825)
Exponential power ($x\geq 0$)	$\exp[(\lambda x)^{\beta}]-1$	$\beta\lambda\exp[(\lambda x)^{\beta}](\lambda x)^{\beta-1}$	1	$[\lambda,\beta]$	Smith and Bain (1975)
Chen ($x\geq 0$)	$\exp(x^b)-1$	$bx^{b-1}\exp(x^b)$	α	b	Chen (2000)
Pham ($x\geq 0$)	$(a^x)^{\beta}-1$	$\beta(a^x)^{\beta}\log(a)$	1	$[a,\beta]$	Pham (2002)

the time to the failure of the device due to the ith defect, for $i\geq 1$. If we assume that the X_i's are iid EW random variables independent of N, which follows a power series distribution (truncated at zero), then the time to the first failure is appropriately modeled by the EWPS distribution.

2. Reliability. From the stochastic representations $X=\min\{X_i\}_{i=1}^N$ and $Z=\max\{X_i\}_{i=1}^N$, we note that the EWPS model can arises in series and parallel systems with identical components, which appear in many industrial applications and biological organisms.
3. Time to relapse of cancer under the first-activation scheme. Here N is the number of carcinogenic cells for an individual left active after the initial treatment and X_i is the time spent for the ith carcinogenic cell to produce a detectable cancer mass, for $i\geq 1$. Assuming that $\{X_i\}_{i\geq 1}$ is a sequence of iid EW random variables independent of N, which follows a power series distribution (truncated at zero), we have that the time to relapse of cancer of a susceptible individual can be modeled by the EWPS class of distributions.
4. Last-activation scheme. As discussed by Cooner et al. (2007), the first activation scheme may be questioned by certain diseases. Let N be the number of latent factors that must all be active by failure and X_i be the time of resistance to a disease manifestation due to the ith latent factor. In the last-activation scheme, it is assumed that failure occurs after all N factors have been active. So, if the X_i's are iid EW random variables independent of N, where N follows a zero-truncated power series distribution, the EWPS class can be able for modeling the time to the failure under the last-activation scheme.

Hereafter, the random variable X following (4) with parameters θ and α and vector of parameters $\boldsymbol{\xi}$ is denoted by $X\sim\text{EWPS}(\theta,\alpha,\boldsymbol{\xi})$. Eq. (4) extends several distributions which have been studied in the literature. The EG distribution (Adamidis and Loukas, 1998) is obtained by taking $H(x;\ \boldsymbol{\xi})=x$ and $C(\theta)=\theta(1-\theta)^{-1}$ with $\theta\in(0,1)$. Further, for $H(x;\ \boldsymbol{\xi})=x$, we obtain the EP (Kus, 2007) and EL (Tahmasbi and Rezaei, 2008) distributions by taking $C(\theta)=e^{\theta}-1$, $\theta>0$, and $C(\theta)=-\log(1-\theta)$, $\theta\in(0,1)$, respectively. In the same way, for $H(x;\ \boldsymbol{\xi})=x^{\gamma}$, we obtain the WG (Barreto-Souza and Cribari-Neto, 2009) and WP (Lu and Shi, 2011) distributions. The EPS distributions come from (4) by combining $H(x;\ \boldsymbol{\xi})=x$ with any $C(\theta)$ listed in Table 1 (see Chahkandi and Ganjali, 2009). Finally, we obtain the WPS distributions from (4) by compounding $H(x;\ \boldsymbol{\xi})=x^{\gamma}$ with any $C(\theta)$ in Table 1 (see Morais and Barreto-Souza, 2011). Table 2 displays some useful quantities and corresponding parameter vectors for special distributions.

3. General properties

3.1. Density, survival and hazard functions

The density function associated to (4) is given by

$$f(x;\theta,\alpha,\boldsymbol{\xi}) = \theta\,\alpha\,h(x;\boldsymbol{\xi})\,\mathrm{e}^{-\alpha H(x;\,\boldsymbol{\xi})}\,\frac{C'(\theta\,\mathrm{e}^{-\alpha H(x;\,\boldsymbol{\xi})})}{C(\theta)}, \quad x>0. \tag{5}$$

Proposition 1. *The EW class of distributions with parameters $c\alpha$ and $\boldsymbol{\xi}$ is a limiting special case of the EWPS class of distributions when $\theta \to 0^+$, where $c = \min\{n \in \mathbb{N} : a_n > 0\}$.*

Proof. This proof uses a similar argument given by Morais and Barreto-Souza (2011). Define $c = \min\{n \in \mathbb{N} : a_n > 0\}$. For $x > 0$, we have

$$\begin{aligned}\lim_{\theta\to 0^+} F(x) &= 1 - \lim_{\theta\to 0^+}\frac{\sum_{n=c}^{\infty} a_n\left(\theta\,\mathrm{e}^{-\alpha H(x;\boldsymbol{\xi})}\right)^n}{\sum_{n=c}^{\infty} a_n\,\theta^n}\\
&= 1 - \lim_{\theta\to 0^+}\frac{\mathrm{e}^{-c\alpha H(x;\boldsymbol{\xi})} + a_c^{-1}\sum_{n=c+1}^{\infty} a_n\,\theta^{n-c}\mathrm{e}^{-n\alpha H(x;\boldsymbol{\xi})}}{1 + a_c^{-1}\sum_{n=c+1}^{\infty} a_n\,\theta^{n-c}}\\
&= 1 - \mathrm{e}^{-c\alpha H(x;\boldsymbol{\xi})}. \quad \square\end{aligned}$$

We now provide an interesting expansion for (5). We have $C'(\theta) = \sum_{n=1}^{\infty} n\,a_n\,\theta^{n-1}$. By using this result in (5), we obtain

$$f(x;\theta,\alpha,\boldsymbol{\xi}) = \sum_{n=1}^{\infty} p_n\,g(x;\; n\alpha,\xi), \tag{6}$$

where $g(x;\; n\alpha,\boldsymbol{\xi})$ is given by (2). Based on Eq. (6), we obtain

$$F(x;\theta,\alpha,\boldsymbol{\xi}) = 1 - \sum_{n=1}^{\infty} p_n\,\mathrm{e}^{-n\alpha H(x;\,\boldsymbol{\xi})}.$$

Hence, the EWPS density function is an infinite mixture of EW densities. So, some mathematical quantities (such as ordinary and incomplete moments, generating function and mean deviations) of the EWPS distributions can be obtained by knowing those quantities for the baseline density function $g(x;\; n\alpha,\boldsymbol{\xi})$. The EWPS survival function becomes

$$S(x;\theta,\alpha,\boldsymbol{\xi}) = \frac{C(\theta\,\mathrm{e}^{-\alpha H(x;\,\boldsymbol{\xi})})}{C(\theta)} \tag{7}$$

and the corresponding hazard rate function reduces to

$$\tau(x;\theta,\alpha,\boldsymbol{\xi}) = \theta\alpha\,h(x;\;\boldsymbol{\xi})\,\mathrm{e}^{-n\alpha H(x;\boldsymbol{\xi})}\,\frac{C'(\theta\,\mathrm{e}^{-\alpha H(x;\,\boldsymbol{\xi})})}{C(\theta\,\mathrm{e}^{-\alpha H(x;\,\boldsymbol{\xi})})}.$$

3.2. Quantiles, moments and order statistics

The EWPS distribution is easily simulated from (4) as follows: if U has a uniform $U(0,1)$ distribution, the solution of the nonlinear equation

$$X = H^{-1}\left\{-\frac{1}{\alpha}\log\left[\frac{C^{-1}(C(\theta)(1-U))}{\theta}\right]\right\}$$

has the EWPS $(\theta,\alpha,\boldsymbol{\xi})$ distribution, where $H^{-1}(\cdot)$ and $C^{-1}(\cdot)$ are the inverse functions of $H(\cdot)$ and $C(\cdot)$, respectively. To simulate data from this nonlinear equation, we can use the matrix programming language Ox through *SolveNLE* subroutine (see Doornik, 2007).

Many of the important characteristics and features of a distribution are obtained through the moment generating function (mgf) and moments. The rth raw moment of X can be determined from (6) and the monotone convergence theorem. So, for $r \in \mathbb{N}$, we obtain

$$\mathrm{E}(X^r) = \sum_{n=1}^{\infty} p_n \, \mathrm{E}(Z^r).$$

Hereafter, Z denotes a random variable with density function $g(z; n\alpha, \boldsymbol{\xi})$.

The incomplete moments and mgf of X can be determined from (6) using the monotone convergence theorem:

$$I_X(y) = \int_0^y x^r f(x)dx = \sum_{n=1}^{\infty} p_n \, I_Z(y)$$

and

$$M_X(t) = \sum_{n=1}^{\infty} p_n \, \mathrm{E}\left(\mathrm{e}^{tZ}\right).$$

Order statistics are among the most fundamental tools in non-parametric statistics and inference. They enter in problems of estimation and hypothesis tests in a variety of ways. Then, we now discuss some properties of the order statistics for the proposed class of distributions. The pdf $f_{i:m}(x)$ of the ith order statistic from a random sample $X_1, \ldots, X_m$ having density function (5) is given by

$$f_{i:m}(x) = \frac{m!}{(i-1)!(m-i)!} f(x; \theta, \alpha, \boldsymbol{\xi}) \left[1 - \frac{C(\theta\,\mathrm{e}^{-\alpha H(x;\,\boldsymbol{\xi})})}{C(\theta)}\right]^{i-1} \left[\frac{C(\theta \mathrm{e}^{-\alpha H(x;\,\boldsymbol{\xi})})}{C(\theta)}\right]^{m-i}, \quad x > 0. \tag{8}$$

By using the binomial expansion, we can write (8) as

$$f_{i:m}(x) = \frac{m!}{(i-1)!(m-i)!} f(x; \theta, \alpha, \boldsymbol{\xi}) \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} S(x; \theta, \alpha, \boldsymbol{\xi})^{m+j-i},$$

where $S(x; \theta, \alpha, \boldsymbol{\xi})$ is given by (7). The corresponding cumulative function becomes

$$F_{i:m}(x) = \sum_{j=0}^{\infty} \sum_{k=i}^{m} (-1)^j \binom{k}{j} \binom{m}{k} S(x; \theta, \alpha, \boldsymbol{\xi})^{m+j-k}.$$

An alternative form for (8) can be obtained from (6) as

$$f_{i:m}(x) = \frac{m!}{(i-1)!(m-i)!} \sum_{n=1}^{\infty} \sum_{j=0}^{i-1} \omega_j \, p_n \, g(x; n\alpha, \boldsymbol{\xi}) S(x; \theta, \alpha, \boldsymbol{\xi})^{m+j-1}, \tag{9}$$

where $\omega_j = (-1)^j \binom{i-1}{j}$. So, the sth raw moment $X_{i:m}$ comes immediately from the above equation

$$\mathrm{E}\left(X_{i:m}^s\right) = \frac{m!}{(i-1)!(m-i)!} \sum_{n=1}^{\infty} \sum_{j=0}^{i-1} \omega_j \, p_n \, \mathrm{E}\left[Z^s \, S(Z; \theta, \alpha, \boldsymbol{\xi})^{m+j-i}\right]. \tag{10}$$

3.3. Reliability and average lifetime

In the context of reliability, the stress–strength model describes the life of a component which has a random strength X subjected to a random stress Y. The component fails at the instant that the stress applied to it exceeds the strength, and the component will function satisfactorily whenever $X > Y$. Hence, $R = \mathrm{P}(X > Y)$ is a measure of component reliability. It has many applications especially in engineering concept. Here, we obtain the form for the reliability R when X and Y are independent random variables having the same EWPS distribution. The quantity R can be expressed as

$$R = \int_0^{\infty} f(x; \theta, \alpha, \boldsymbol{\xi}) F(x; \theta, \alpha, \boldsymbol{\xi}) dx. \tag{11}$$

Substituting (4) and (5) into Eq. (11), we obtain

$$\begin{aligned} R &= \int_0^{\infty} \theta\, \alpha\, h(x; \boldsymbol{\xi})\, \mathrm{e}^{-\alpha H(x;\,\boldsymbol{\xi})} \frac{C'(\theta \mathrm{e}^{-\alpha H(x;\,\boldsymbol{\xi})})}{C(\theta)} \left[1 - \frac{C(\theta \mathrm{e}^{-\alpha H(x;\boldsymbol{\xi})})}{C(\theta)}\right] dx \\ &= 1 - \sum_{n=1}^{\infty} p_n \int_0^{\infty} g(x; n\alpha, \boldsymbol{\xi}) S(x; \theta, \alpha, \boldsymbol{\xi}) dx, \end{aligned}$$

where the integral can be determined from the baseline EW distribution.

The average lifetime is given by

$$t_m = \sum_{n=1}^{\infty} p_n \int_0^{\infty} \mathrm{e}^{-n\alpha H(x;\,\boldsymbol{\xi})} dx.$$

In fields such as actuarial sciences, survival studies and reliability theory, the concept of mean residual life has been of much interest; see a survey by Guess and Proschan (1985). Given that there was no failure prior to x_0, the residual life is the period from time x_0 until the time of failure. The mean residual lifetime can be expressed as

$$\begin{aligned} m(x_0; \theta, \alpha, \boldsymbol{\xi}) &= [\Pr(X > x_0)]^{-1} \int_0^{\infty} y f(x_0 + y; \theta, \alpha, \boldsymbol{\xi}) dy \\ &= [S(x_0)]^{-1} \sum_{n=1}^{\infty} p_n \int_0^{\infty} y\, g(x_0 + y; n\alpha, \boldsymbol{\xi}) dy. \end{aligned}$$

The last integral can be computed from the baseline EW distribution. Further, $m(x_0; \theta, \alpha, \boldsymbol{\xi}) \to \mathrm{E}(X)$ as $x_0 \to 0$. Some results of this section can be obtained numerically in any symbolic software such as MAPLE (Garvan, 2002), MATLAB (Sigmon and Davis, 2002), MATHEMATICA (Wolfram, 2003), Ox (Doornik, 2007) and R (R Development Core Team, 2009). The Ox (for academic purposes) and R are freely distributed and available at http://www.doornik.com and http://www.r-project.org, respectively. The results are easily computed by taking in these sums a large positive integer value in place of ∞.

4. Maximum likelihood estimation

Here, we determine the maximum likelihood estimates (MLEs) of the parameters of the EWPS class of distributions from complete samples only. Let $x_1, \ldots, x_n$ be observed values from the EWPS distribution with parameters θ, α and $\boldsymbol{\xi}$. Let $\Theta = (\theta, \alpha, \boldsymbol{\xi})^\top$ be the $p \times 1$ parameter vector. The total log-likelihood function for Θ is given by

$$\ell_n = \ell_n(\Theta) = n[\log(\theta) + \log(\alpha) - \log(C(\theta))] - \alpha \sum_{i=1}^{n} H(x_i;\ \boldsymbol{\xi}) + \sum_{i=1}^{n} \log[h(x_i;\ \boldsymbol{\xi})] + \sum_{i=1}^{n} \log[C'(\theta\, \mathrm{e}^{-\alpha H(x_i;\,\boldsymbol{\xi})})]. \quad (12)$$

The log-likelihood can be maximized either directly by using the SAS (PROC NLMIXED) or the Ox program (sub-routine MaxBFGS) (see Doornik, 2007) or by solving the nonlinear likelihood equations obtained by differentiating (12). The components of the score function $U_n(\Theta) = (\partial\ell_n/\partial\theta, \partial\ell_n/\partial\alpha, \partial\ell_n/\partial\xi)^\top$ are

$$\frac{\partial \ell_n}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} H(x_i;\ \boldsymbol{\xi}) - \theta \sum_{i=1}^{n} H(x_i;\ \boldsymbol{\xi}) \mathrm{e}^{-\alpha H(x_i;\,\boldsymbol{\xi})} \frac{C''(\theta\, \mathrm{e}^{-\alpha H(x_i;\,\boldsymbol{\xi})})}{C'(\theta\, \mathrm{e}^{-\alpha H(x_i;\,\boldsymbol{\xi})})},$$

$$\frac{\partial \ell_n}{\partial \theta} = \frac{n}{\theta} - n\frac{C'(\theta)}{C(\theta)} + \sum_{i=1}^{n} \mathrm{e}^{-\alpha H(x_i;\,\boldsymbol{\xi})} \frac{C''(\theta\, \mathrm{e}^{-\alpha H(x_i;\,\boldsymbol{\xi})})}{C'(\theta\, \mathrm{e}^{-\alpha H(x_i;\,\boldsymbol{\xi})})}$$

and

$$\frac{\partial \ell_n}{\partial \boldsymbol{\xi}_k} = \sum_{i=1}^{n} \frac{\partial \log h(x_i;\, \xi)}{\partial \xi_k} - \alpha \sum_{i=1}^{n} \frac{\partial H(x_i;\, \xi)}{\partial \xi_k} \left[1 + \theta \mathrm{e}^{-\alpha H(x_i;\,\xi)} \frac{C''(\theta\, \mathrm{e}^{-\alpha H(x_i;\,\xi)})}{C'(\theta\, \mathrm{e}^{-\alpha H(x_i;\,\xi)})}\right].$$

For interval estimation on the model parameters, we require the observed information matrix

$$J_n(\Theta) = -\begin{pmatrix} U_{\theta\theta} & U_{\theta\alpha} & | & U_{\theta\xi}^\top \\ U_{\alpha\theta} & U_{\alpha\alpha} & | & U_{\alpha\xi}^\top \\ - & - & - & - \\ U_{\theta\xi} & U_{\alpha\xi} & | & U_{\xi\xi} \end{pmatrix},$$

whose elements are listed in Appendix. Let $\widehat{\Theta}$ be the MLE of Θ. Under standard regular conditions (Cox and Hinkley, 1974) that are fulfilled for the proposed model whenever the parameters are in the interior of the parameter space, we can approximate the distribution of $\sqrt{n}(\widehat{\Theta} - \Theta)$ by the multivariate normal $N_p(0, K(\Theta)^{-1})$, where $K(\Theta) = \lim_{n\to\infty} J_n(\Theta)$ is the unit information matrix and p is the number of parameters of the compounded distribution.

Often with lifetime data and reliability studies, one encounters censoring. A very simple random censoring mechanism very often realistic is one in which each individual i is assumed to have a lifetime X_i and a censoring time C_i, where X_i and C_i are independent random variables. Suppose that the data consist of n independent observations $x_i = \min(X_i, C_i)$ and

$\delta_i = I(X_i \le C_i)$ is such that $\delta_i = 1$ if X_i is a time to event and $\delta_i = 0$ if it is right censored for $i = 1, \ldots, n$. The censored likelihood $L(\Theta)$ for the model parameters is

$$L(\Theta) \propto \prod_{i=1}^{n}[f(x_i;\theta,\alpha,\boldsymbol{\xi})]^{\delta_i}\,[S(x_i;\theta,\alpha,\boldsymbol{\xi})]^{1-\delta_i},$$

where $f(x;\theta,\alpha,\boldsymbol{\xi})$ and $S(x;\theta,\alpha,\boldsymbol{\xi})$ are given in (5) and (7), respectively.

5. Maximum entropy identification

The concept of Shannon entropy is the central role of information theory sometimes referred as measure of uncertainty. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty. Shannon (1948) introduced the probabilistic definition of entropy which is closely connected with the definition of entropy in statistical mechanics. Let X be a random variable of a continuous distribution with density f. Then, the Shannon entropy of X is defined by

$$\mathbb{H}_{Sh}(f) = -\int_{\mathbb{R}} f(x;\theta,\alpha,\boldsymbol{\xi})\log[f(x;\theta,\alpha,\boldsymbol{\xi})]\,dx. \tag{13}$$

Jaynes (1957) introduced one of the most powerful techniques employed in the field of probability and statistics called the "*maximum entropy method*". This method is closely related to the Shannon entropy and considers a class of density functions

$$\mathbb{F} = \left\{f(x;\theta,\alpha,\boldsymbol{\xi}) : \mathrm{E}_f(T_i(X)) = \alpha_i,\ i = 0,\ldots,m\right\},$$

where $T_1(X),\ldots,T_m(X)$ are absolutely integrable functions with respect to f, and $T_0(X) = a_0 = 1$. In the continuous case, the maximum entropy principle suggests deriving the unknown density function of the random variable X by the model that maximizes the Shannon entropy in (13), subject to the information constraints defined in the class $\mathbb{F}$. Shore and Johnson (1980) treated the maximum entropy method axiomatically. This method has been successfully applied in a wide variety of fields and has also been used for the characterization of several standard probability distributions; see, for example, Kapur (1989), Soofi (2000) and Zografos and Balakrishnan (2009).

The maximum entropy distribution is the density of the class $\mathbb{F}$, denoted by f^{ME}, determined as the solution of the optimization problem

$$f^{ME}(x;\theta,\alpha,\boldsymbol{\xi}) = \arg\max_{f\in\mathbb{F}}\mathbb{H}_{Sh}.$$

Jaynes (1957, p. 623) states that the maximum entropy distribution f^{ME} obtained by the constrained maximization problem described above, "*is the only unbiased assignment we can make; to use any other would amount to arbitrary assumption of information which by hypothesis we do not have*". It is the distribution which should not incorporate additional exterior information other than which is specified by the constraints. We now derive suitable constraints in order to provide a maximum entropy characterization for the class (4). For this purpose, the next result plays an important role.

Proposition 2. *Let X be a random variable with pdf given by (5). Then,*

C1. $\mathrm{E}\left\{\log[C'(\theta\,\mathrm{e}^{-\alpha H(X;\,\boldsymbol{\xi})})]\right\} = \frac{\theta}{C(\theta)}\mathrm{E}\left\{C'(\theta\,\mathrm{e}^{-\alpha H(Y;\,\boldsymbol{\xi})})\log[C'(\theta\,\mathrm{e}^{-\alpha H(Y;\,\boldsymbol{\xi})})]\right\};$

C2. $\mathrm{E}\left\{\log[h(X;\ \boldsymbol{\xi})]\right\} = \frac{\theta}{C(\theta)}\mathrm{E}\left\{C'(\theta\,\mathrm{e}^{-\alpha H(Y;\,\boldsymbol{\xi})})\log[h(Y;\ \boldsymbol{\xi})]\right\};$

C3. $\mathrm{E}\left[H(X;\ \boldsymbol{\xi})\right] = \frac{\theta}{C(\theta)}\mathrm{E}\left[C'(\theta\,\mathrm{e}^{-\alpha H(Y;\,\boldsymbol{\xi})})H(Y;\ \boldsymbol{\xi})\right],$

where Y follows the EW distribution with density function (2).

Proof. The constraints C1, C2 and C3 are easily demonstrated and then the proofs are omitted. □

The next proposition reveals that the EWPS distribution has maximum entropy in the class of all probability distributions specified by the constraints stated in the previous proposition.

Proposition 3. *The pdf f of a random variable X, given by (5), is the unique solution of the optimization problem*

$$f(x;\theta,\alpha,\boldsymbol{\xi}) = \arg\max_{h}\mathbb{H}_{Sh},$$

under the constraints C1, C2 *and* C3 *presented in the Proposition 2.*

Proof. Let τ be a pdf that satisfies the constraints C1, C2 and C3. The Kullback–Leibler divergence between τ and f is

$$D(\tau, f) = \int_{\mathbb{R}} \tau(x; \theta, \alpha, \boldsymbol{\xi}) \log\left[\frac{\tau(x; \theta, \alpha, \boldsymbol{\xi})}{f(x; \theta, \alpha, \boldsymbol{\xi})}\right] dx.$$

Following Cover and Thomas (1991), we obtain

$$\begin{aligned} 0 \le D(\tau, f) &= \int_{\mathbb{R}} \tau(x; \theta, \alpha, \boldsymbol{\xi}) \log[\tau(x; \theta, \alpha, \boldsymbol{\xi})]\, dx - \int_{\mathbb{R}} \tau(x; \theta, \alpha, \boldsymbol{\xi}) \log[f(x; \theta, \alpha, \boldsymbol{\xi})]\, dx \\ &= -\mathbb{H}_{Sh}(\tau; \theta, \alpha, \boldsymbol{\xi}) - \int_{\mathbb{R}} \tau(x; \theta, \alpha, \boldsymbol{\xi}) \log[f(x; \theta, \alpha, \boldsymbol{\xi})]\, dx. \end{aligned}$$

From the definition of f and based on the constraints C1, C2 and C3, we have

$$\begin{aligned} \int_{\mathbb{R}} \tau(x) \log[f(x)]\, dx &= \log(\theta\alpha) + \frac{\theta}{C(\theta)} \mathrm{E}\left\{C'(\theta\, \mathrm{e}^{-\alpha H(Y;\, \boldsymbol{\xi})}) \log[h(Y;\ \boldsymbol{\xi})]\right\} - \log[C(\theta)] \\ &\quad - \alpha \frac{\theta}{C(\theta)} \mathrm{E}\left[C'(\theta\, \mathrm{e}^{-\alpha H(Y;\, \boldsymbol{\xi})}) H(Y;\ \boldsymbol{\xi})\right] \\ &\quad + \frac{\theta}{C(\theta)} \mathrm{E}\left\{\log\left[C'(\theta\, \mathrm{e}^{-\alpha H(Y;\, \boldsymbol{\xi})})\right] C'(\theta\, \mathrm{e}^{-\alpha H(Y;\, \boldsymbol{\xi})})\right\} \\ &= \int_{\mathbb{R}} f(x; \theta, \alpha, \boldsymbol{\xi}) \log[f(x; \theta, \alpha, \boldsymbol{\xi})]\, dx = -\mathbb{H}_{Sh}(f), \end{aligned}$$

where Y is defined as before. So, we obtain $\mathbb{H}_{Sh}(\tau) \le \mathbb{H}_{Sh}(f)$ with equality if and only if $\tau(x; \theta, \alpha, \boldsymbol{\xi}) = f(x; \theta, \alpha, \boldsymbol{\xi})$ for all x, except for a null measure set, thus proving the uniqueness. □

The intermediate steps in the above proof in fact provide the following explicit expression for the Shannon entropy of X

$$\begin{aligned} \mathbb{H}_{Sh}(f) &= -\log(\theta\alpha) - \frac{\theta}{C(\theta)} \mathrm{E}\left\{C'(\theta\, \mathrm{e}^{-\alpha H(Y;\, \boldsymbol{\xi})}) \log[h(Y;\ \boldsymbol{\xi})]\right\} + \log[C(\theta)] \\ &\quad + \alpha \frac{\theta}{C(\theta)} \mathrm{E}\left[C'(\theta\, \mathrm{e}^{-\alpha H(Y;\, \boldsymbol{\xi})}) H(Y;\ \boldsymbol{\xi})\right] - \frac{\theta}{C(\theta)} \mathrm{E}\left\{C'(\theta\, \mathrm{e}^{-\alpha H(Y;\, \boldsymbol{\xi})}) \log\left[C'(\theta\, \mathrm{e}^{-\alpha H(Y;\, \boldsymbol{\xi})})\right]\right\}. \end{aligned}$$

For some EWPS distributions, the above results can only be obtained numerically.

6. Special models

In this section, we study three special models of the EWPS class of distributions. We provide plots of the density and hazard rate functions for selected parameter values to illustrate the flexibility of these distributions. We offer some explicit expressions for the moments and moments of the order statistics.

6.1. Modified Weibull geometric distribution

The modified Weibull geometric (MWG) distribution is defined by the cdf (4) with $H(x;\ \boldsymbol{\xi}) = x^{\gamma}$ and $C(\theta) = \theta(1-\theta)^{-1}$ leading to

$$F(x; \theta, \alpha, \gamma, \lambda) = 1 - \frac{(1-\theta)\exp\left(-\alpha x^{\gamma} \mathrm{e}^{\lambda x}\right)}{1 - \theta \exp\left(-\alpha x^{\gamma} \mathrm{e}^{\lambda x}\right)}, \quad x > 0,$$

where $\theta \in (0, 1)$. The associated pdf and hazard rate function are

$$f(x; \theta, \alpha, \gamma, \lambda) = \alpha(1-\theta)(\gamma + \lambda x)\, x^{\gamma-1} \frac{\exp\left(\lambda x - \alpha x^{\gamma} \mathrm{e}^{\lambda x}\right)}{\left[1 - \theta \exp\left(-\alpha x^{\gamma} \mathrm{e}^{\lambda x}\right)\right]^2}$$

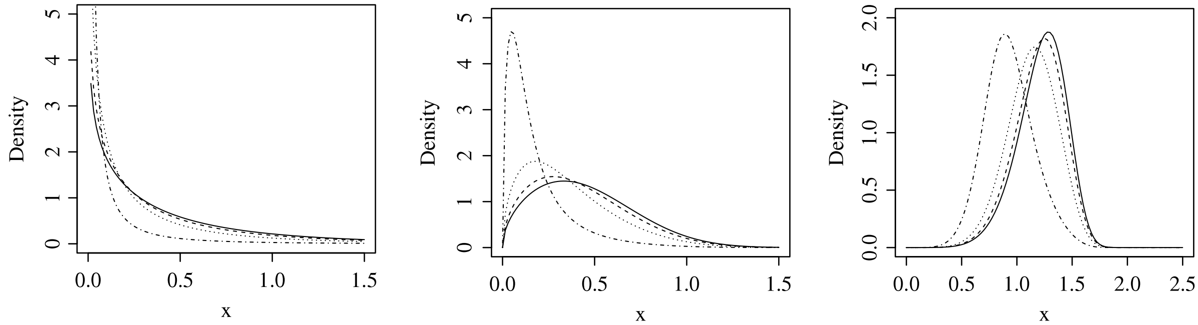
and

$$\tau(x; \theta, \alpha, \gamma, \lambda) = \alpha(\gamma + \lambda x)\, x^{\gamma-1} \frac{\exp(\lambda x)}{1-\theta \exp\left(-\alpha x^{\gamma} \mathrm{e}^{\lambda x}\right)},$$

respectively. The MWG distribution includes the WG distribution (Barreto-Souza et al., 2010) when $\lambda = 0$. Further, for $\lambda = 0$ and $\alpha = 1$, we obtain the EG distribution (Adamidis and Loukas, 1998). Figs. 1 and 2 display the density and hazard functions of the MWG distribution for selected parameter values.

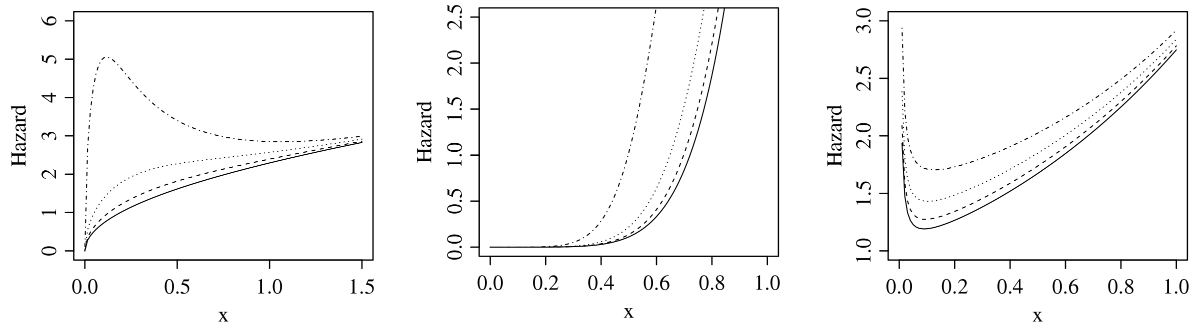
The rth raw moment of the random variable X having the MWG distribution is determined in closed-form from (6) as

$$\mathrm{E}(X^r) = \sum_{n=1}^{\infty} p_n\, \mu_r(n), \tag{14}$$



(a) $\alpha = 2,\ \gamma = 1.5$ and $\lambda = 0.5$. (b) $\alpha = 2,\ \gamma = 0.8$ and $\lambda = 0.01$. (c) $\alpha = 0.1,\ \gamma = 6$ and $\lambda = 0.5$.

Fig. 1. Plots of the MWG density function for $\theta = 0.01$ (solid line), $\theta = 0.2$ (dashed line), $\theta = 0.5$ (dotted line) and $\theta = 0.9$ (dotdash line).



(a) $\alpha = 2,\ \gamma = 1.5$ and $\lambda = 0.5$. (b) $\alpha = 2,\ \gamma = 0.8$ and $\lambda = 0.01$. (c) $\alpha = 0.1,\ \gamma = 6$ and $\lambda = 0.5$.

Fig. 2. Plots of the MWG hazard rate function for $\theta = 0.01$ (solid line), $\theta = 0.2$ (dashed line), $\theta = 0.5$ (dotted line) and $\theta = 0.9$ (dotdash line).

where $\mu_r(n) = \int_0^\infty x^r g(x;\ n\alpha, \gamma, \lambda)dx$ denotes the rth raw moment of the MW distribution with parameters $n\alpha$, γ and λ. Here, p_n corresponds to the geometric probability function. Carrasco et al. (2008) obtained an infinite representation for the rth raw moment of the MW distribution with these parameters given by

$$\mu_r(n) = \sum_{i_1,\ldots,i_r=1}^{\infty} \frac{A_{i_1,\ldots,i_r}\,\Gamma(s_r/\gamma + 1)}{(n\alpha)^{s_r/\gamma}}, \tag{15}$$

where

$$A_{i_1,\ldots,i_r} = a_{i_1}, \ldots, a_{i_r} \quad \text{and} \quad s_r = i_1, \ldots, i_r,$$

and

$$a_i = \frac{(-1)^{i+1} i^{i-2}}{(i-1)!} \left(\frac{\lambda}{\gamma}\right)^{i-1}.$$

Hence, the ordinary moments of X can be obtained directly from Eqs. (14) and (15).

The density of the ith order statistic $X_{i:m}$ in a random sample of size m from the MWG distribution is given by (for $i = 1, \ldots, m$)

$$f_{i:m}(x) = \frac{m!}{(i-1)!(m-i)!} \sum_{n=1}^{\infty} \sum_{j=0}^{i-1} \omega_j\, p_n \left[\frac{(1-\theta)\exp\left(-\alpha x^{\gamma} e^{\lambda x}\right)}{1-\theta\exp\left(-\alpha x^{\gamma} e^{\lambda x}\right)}\right]^{m+j-i} g(x;\, n\alpha, \gamma, \lambda),$$

where $g(x;\, n\alpha, \gamma, \lambda)$ denotes the MW density function with parameters $n\alpha$, γ and λ. From (10), we obtain

$$\mathrm{E}\left(X_{i:m}^s\right) = \frac{m!}{(i-1)!(m-i)!} \sum_{n=1}^{\infty} \sum_{j=0}^{i-1} \omega_j\, p_n\, \mathrm{E}\left\{X^s \left[\frac{(1-\theta)\exp\left(-\alpha X^{\gamma} e^{\lambda X}\right)}{1-\theta\exp\left(-\alpha X^{\gamma} e^{\lambda X}\right)}\right]^{m+j-i}\right\}.$$

6.2. Pareto Poisson distribution

The Pareto Poisson (PP) distribution is defined by taking $H(x;\ \boldsymbol{\xi}) = \log(x/k)$ and $C(\theta) = \mathrm{e}^{\theta} - 1$ in (4) leading to

$$F(x;\theta,\alpha,k) = 1 - \frac{\exp\left[\theta\,(k/x)^{\alpha}\right] - 1}{\mathrm{e}^{\theta} - 1}, \quad x \geq k.$$

The corresponding pdf and hazard rate function are

$$f(x;\theta,\alpha,k) = \frac{\theta\,\alpha\,k^{\alpha}\exp\left[\theta\,(k/x)^{\alpha}\right]}{(\mathrm{e}^{\theta} - 1)\,x^{\alpha+1}}$$

and

$$\tau(x;\theta,\alpha,k) = \frac{\theta\,\alpha\,k^{\alpha}\exp\left[\theta\,(k/x)^{\alpha}\right]}{x^{\alpha+1}\left\{\exp\left[\theta\,(k/x)^{\alpha}\right] - 1\right\}},$$

respectively. We obtain the Pareto distribution as a sub-model when $\theta \to 0$. The rth moment of the random variable X having the PP distribution becomes

$$\mathrm{E}(X^{r}) = \frac{\alpha k^{r}}{(\mathrm{e}^{\theta} - 1)}\sum_{n=1}^{\infty}\frac{\theta^{n}}{(n-1)!\,(n\alpha - r)}, \quad n\alpha > r. \tag{16}$$

In particular, setting $r = 1$ in (16), the mean of X reduces to

$$\mu = \frac{\alpha k}{\mathrm{e}^{\theta} - 1}\sum_{n=1}^{\infty}\frac{\theta^{n}}{(n-1)!\,(n\alpha - 1)}, \quad n\alpha > 1.$$

From Eq. (10), the sth moment of the ith order statistic (for $i = 1, \ldots, m$) is given by

$$\mathrm{E}\left(X_{i:m}^{s}\right) = \frac{m!}{(i-1)!(m-i)!}\sum_{n=1}^{\infty}\sum_{j=0}^{i-1}\omega_j\,p_n\,\mathrm{E}\left[X^{s}\left(\frac{\exp(\theta\,(k/X)^{\alpha}) - 1}{\mathrm{e}^{\theta} - 1}\right)^{m+j-i}\right],$$

where p_n denotes the Poisson probability function. In addition, after some algebra, the Shannon entropy for the PP distribution reduces to

$$\mathbb{H}_{Sh}(f) = \log\left(\frac{\mathrm{e}^{\theta} - 1}{\theta\alpha}\right) - \frac{\theta}{\mathrm{e}^{\theta} - 1}\left(\mu_1 - \alpha\mu_2 + \mu_3\right),$$

where

$$\mu_1 = \mathrm{E}\left[\exp\left\{\theta\left(\frac{k}{X}\right)^{\alpha}\right\}\log\left(\frac{1}{X}\right)\right] = \frac{1}{2(\mathrm{e}^{\theta} - 1)}\left\{\frac{\mathrm{Chi}(2\theta) - \log(2\theta) + \mathrm{Shi}(2\theta) - \gamma}{\alpha} - (\mathrm{e}^{2\theta} - 1)\log k\right\},$$

$$\mu_2 = \mathrm{E}\left[\exp\left\{\theta\left(\frac{k}{X}\right)^{\alpha}\right\}\log\left(\frac{X}{k}\right)\right] = \frac{\mathrm{Chi}(2\theta) - \log(2\theta) + \mathrm{Shi}(2\theta) - \gamma}{2\alpha(\mathrm{e}^{\theta} - 1)}$$

and

$$\mu_3 = \mathrm{E}\left[\theta\exp\left\{\theta\left(\frac{k}{X}\right)^{\alpha}\right\}\left(\frac{k}{X}\right)^{\alpha}\right] = \frac{\alpha\,\theta\,k^{2\alpha}}{4(\mathrm{e}^{\theta} - 1)}\left\{1 - (2\theta + 1)\mathrm{e}^{2\theta}\right\},$$

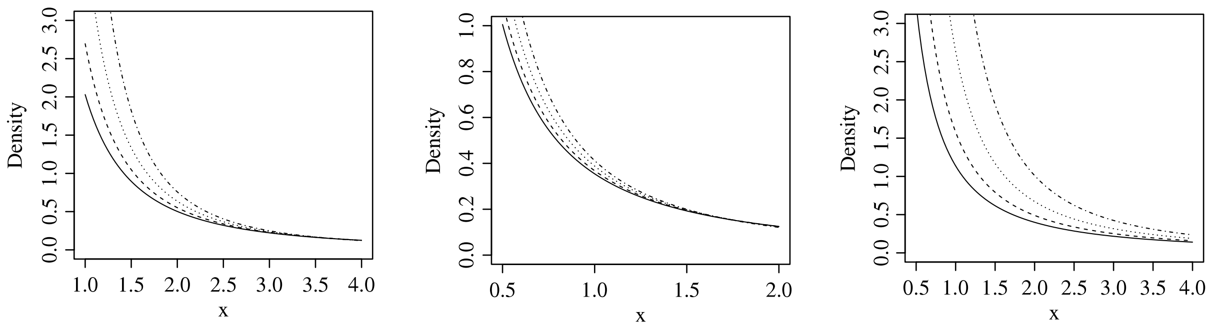
where

$$\mathrm{Chi}(z) = \gamma + \log z + \int_0^z \frac{\cosh(t) - 1}{t}dt$$

is the hyperbolic cosine integral,

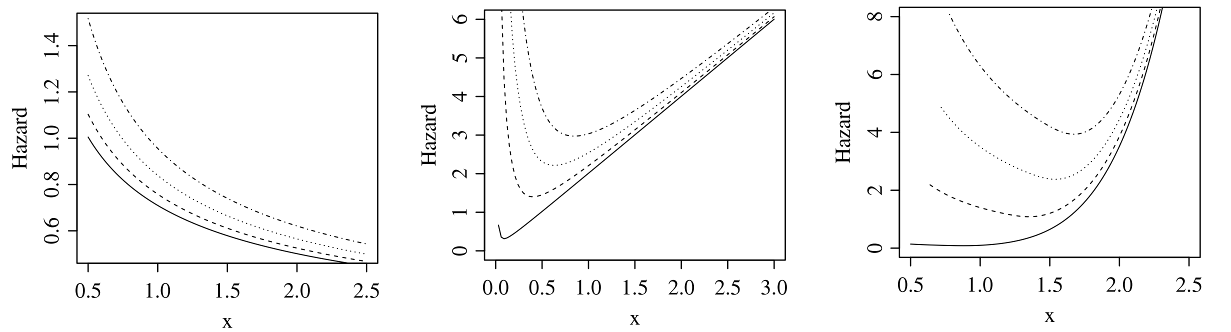
$$\mathrm{Shi}(z) = \int_0^z \frac{\sinh(t) - 1}{t}dt$$

is the hyperbolic sine integral and $\gamma \approx 0.577216$ is the Euler–Mascheroni constant (see Figs. 3 and 4).



(a) $\alpha = 2,\ \gamma = 1.5$ and $\lambda = 0.5$. (b) $\alpha = 2,\ \gamma = 0.8$ and $\lambda = 0.01$. (c) $\alpha = 0.1,\ \gamma = 6$ and $\lambda = 0.5$.

Fig. 3. Plots of the PP density function for $\theta = 0.01$ (solid line), $\theta = 0.2$ (dashed line), $\theta = 0.5$ (dotted line) and $\theta = 0.9$ (dotdash line).



(a) $\alpha = k = 0.5$. (b) $\alpha = 2$ and $k = 1$. (c) $\alpha = 7$ and $k = 2$.

Fig. 4. Plots of the PP hazard rate function for $\theta = 0.01$ (solid line), $\theta = 0.2$ (dashed line), $\theta = 0.5$ (dotted line) and $\theta = 0.9$ (dotdash line).

6.3. Chen logarithmic distribution

The Chen logarithmic (CL) distribution is defined by the cdf (4) with $H(x;\ \boldsymbol{\xi}) = \exp(x^{\beta}) - 1$ and $C(\theta) = -\log(1-\theta)$ leading to

$$F(x) = 1 - \frac{\log\left\{1 - \theta \exp\left[-\alpha(\exp(x^{\beta}) - 1)\right]\right\}}{\log(1-\theta)}, \quad x > 0,$$

where $\theta \in (0, 1)$. The associated pdf and hazard rate function are

$$f(x) = \frac{\theta\, \alpha\, b\, x^{b-1} \exp\left\{x^{b} - \alpha\left[\exp(x^{b}) - 1\right]\right\}}{\log(1-\theta)\left\{\theta \exp\left[-\alpha(\exp(x^{b}) - 1)\right] - 1\right\}}$$

and

$$\tau(x) = \frac{\theta\, \alpha\, b\, x^{b-1} \exp\left[x^{b} - \alpha(\exp(x^{b}) - 1)\right]}{\left\{\theta \exp\left[-\alpha(\exp(x^{b}) - 1)\right] - 1\right\} \log\left\{1 - \theta \exp\left[-\alpha(\exp(x^{b}) - 1)\right]\right\}},$$

respectively.

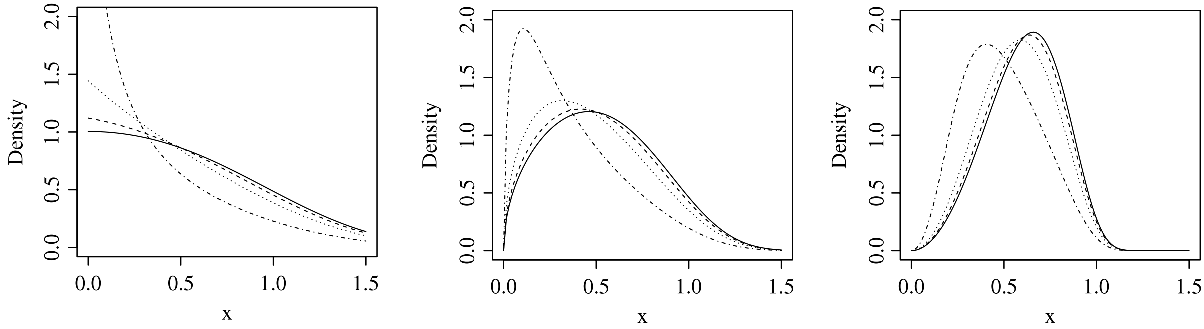
Proposition 1 implies that the Chen distribution is a limiting special case when $\theta \to 0^{+}$.

The density of the ith order statistic $X_{i:m}$ in a random sample of size m from the CL distribution is given by (for $i = 1, \ldots, m$)

$$f_{i:m}(x) = \frac{m!}{(i-1)!(m-i)!} \sum_{n=1}^{\infty} \sum_{j=0}^{i-1} \omega_j^{*}\, p_n\, g(x;\, n\alpha, b) \left\{\log\left[1 - \theta \exp(\alpha - \alpha e^{x^{b}})\right]\right\}^{m+j-1},$$

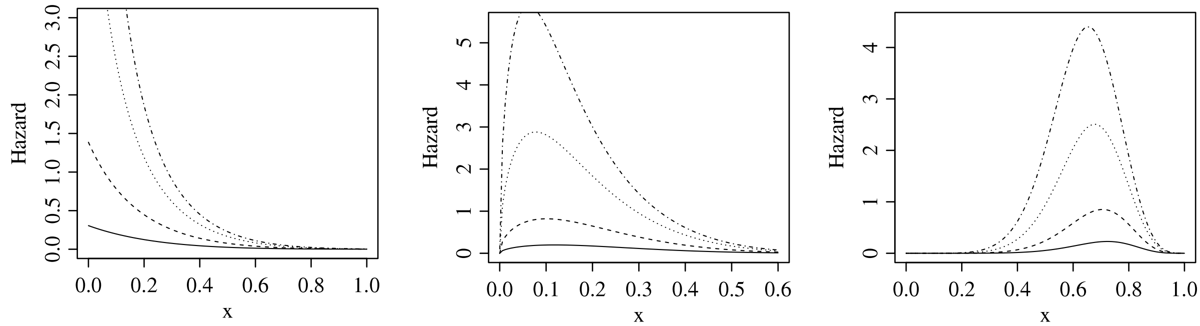
where $g(x;\, n\alpha, b)$ denotes the Chen density function with parameters $n\alpha$ and b and p_n denotes the logarithmic probability function and

$$\omega_j^{*} = (-1)^{j} \binom{i-1}{j} \left[\frac{1}{\log(1-\theta)}\right]^{m+j-1}.$$



(a) $\alpha = b = 1$. (b) $\alpha = b = 1.5$. (c) $\alpha = 2.5$ and $b = 3$.

Fig. 5. Plots of the CL density function for $\theta = 0.01$ (solid line), $\theta = 0.2$ (dashed line), $\theta = 0.5$ (dotted line) and $\theta = 0.9$ (dotdash line).



(a) $\alpha = 2$ and $b = 1$. (b) $\alpha = 3.5$ and $b = 7$. (c) $\alpha = 3.5$ and $b = 1.5$.

Fig. 6. Plots of the CL hazard rate function for $\theta = 0.01$ (solid line), $\theta = 0.2$ (dashed line), $\theta = 0.5$ (dotted line) and $\theta = 0.9$ (dotdash line).

In a similar manner, the sth raw moment of $X_{i:m}$ is obtained directly from

$$\mathrm{E}\left(X_{i:m}^{s}\right) = \frac{m!}{(i-1)!(m-i)!}\sum_{n=1}^{\infty}\sum_{j=0}^{i-1}\omega_j\, p_n\, \mathrm{E}\left\{Z^{s}\exp\left[n\alpha(m+j-1)(1-\exp(Z^{b}))\right]\right\},$$

where $Z \sim \mathrm{Chen}(n\alpha, b)$.

7. Applications

In this section, we compare the results of the fitted special models of the EWPS class by means of two real data sets for illustrative purposes. In order to estimate the parameters of these special models, we adopt the maximum likelihood method (as discussed in Section 4) and all the computations were done using the subroutine `NLMixed` of the SAS software. A good alternative is to use the software **R** for which Nadarajah et al. (2012) introduced the package `Compounding` for dealing with continuous distributions obtained by compounding continuous distributions with discrete distributions. They demonstrated its use by computing values of the cumulative and density functions, quantile and hazard rate functions, generating random samples from a population with compounding distribution, and computing mean and variance of a random variable with a compounding distribution (see Figs. 5 and 6).

First, we consider a data set from Fonseca and França (2007), who studied the soil fertility influence and the characterization of the biologic fixation of N_2 for the *Dimorphandra wilsonii rizz growth*. For 128 plants, they made measures of the phosphorus concentration in the leaves. The data are listed in Table 3. We fit the Gompertz Poisson (GP), Chen Poisson (CP) and CL models to these data. We also fit the three-parameter sub-model WG (Barreto-Souza et al., 2010).

Tables 4 and 5 display some descriptive statistics and the MLEs (with corresponding standard errors in parentheses), the maximized log-likelihood and the Kolmogorov–Smirnov statistic for the fitted models. Since the values of the Akaike information criterion (AIC), Bayesian information criterion (BIC) and consistent Akaike information criterion (CAIC) are smaller for the CL distribution compared with those values of the other models, this new distribution seems to be a very competitive model for these data.

Plots of the pdf and cdf of the WG, GP, CP and CL fitted models to these data are displayed in Fig. 7. They indicate that the CL distribution is superior to the other distributions in terms of model fitting. Based on these plots, we conclude that the CL distribution provides a better fit to these data than the WG, GP and CP models.

Table 3
Phosphorus concentration in leaves data.

0.22	0.17	0.11	0.10	0.15	0.06	0.05	0.07	0.12	0.09	0.23	0.25	0.23
0.24	0.20	0.08	0.11	0.12	0.10	0.06	0.20	0.17	0.20	0.11	0.16	0.09
0.10	0.12	0.12	0.10	0.09	0.17	0.19	0.21	0.18	0.26	0.19	0.17	0.18
0.20	0.24	0.19	0.21	0.22	0.17	0.08	0.08	0.06	0.09	0.22	0.23	0.22
0.19	0.27	0.16	0.28	0.11	0.10	0.20	0.12	0.15	0.08	0.12	0.09	0.14
0.07	0.09	0.05	0.06	0.11	0.16	0.20	0.25	0.16	0.13	0.11	0.11	0.11
0.08	0.22	0.11	0.13	0.12	0.15	0.12	0.11	0.11	0.15	0.10	0.15	0.17
0.14	0.12	0.18	0.14	0.18	0.13	0.12	0.14	0.09	0.10	0.13	0.09	0.11
0.11	0.14	0.07	0.07	0.19	0.17	0.18	0.16	0.19	0.15	0.07	0.09	0.17
0.10	0.08	0.15	0.21	0.16	0.08	0.10	0.06	0.08	0.12	0.13		

Table 4
Descriptive statistics.

Min.	Q_1	Q_2	Mean	Q_3	Max.	Var.
0.0500	0.1000	0.1300	0.1408	0.1800	0.2800	0.0030

Table 5
MLEs of the parameters with corresponding SE's (given in parentheses) and maximized log-likelihoods of the WG, GP, CP and CL models for the first data set. The statistics AIC, BIC and CAIC are also displayed.

Model	$\widehat{\theta}$	$\widehat{\alpha}$	$\widehat{\gamma}$	AIC	BIC	CAIC	K–S	$-2\ell(\widehat{\Theta})$
WG	0.9995	2.4471	4.2041	−378.5	−370.0	−378.3	0.0873	−384.5
	(0.0017)	(8.7059)	(0.3022)					
	$\widehat{\theta}$	$\widehat{\alpha}$	$\widehat{\beta}$					
GP	2.9478	0.3169	19.7047	−368.7	−360.2	−368.5	0.1201	−374.7
	(1.2627)	(0.1473)	(1.6135)					
	$\widehat{\theta}$	$\widehat{\alpha}$	$\widehat{b}$					
CP	15.4386	14.7817	2.9212	−383.7	−375.2	−383.5	0.1159	−389.7
	(22.8318)	(28.1576)	(0.2634)					
CL	0.9999	52232	7.5882	−395.8	−387.2	−395.6	0.0678	−401.8
	(0.0001)	(0.0000)	(0.2039)					

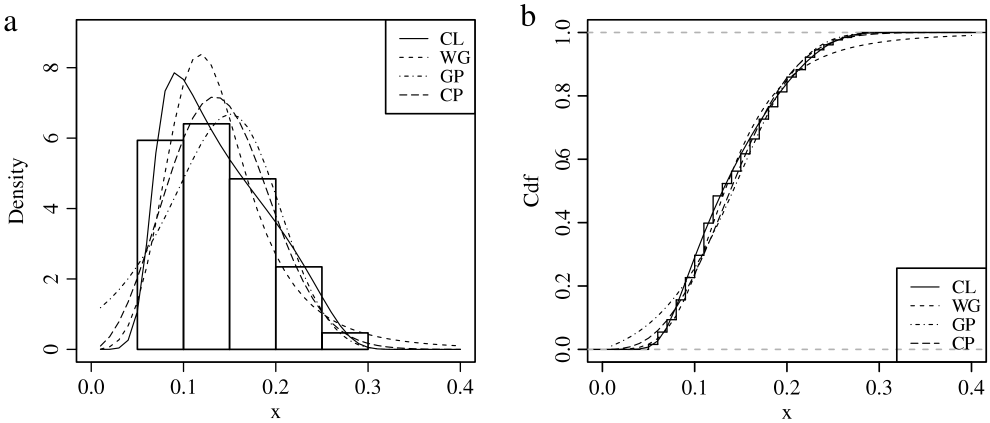


Fig. 7. Estimated (a) pdf and (b) cdf for the CL, CP, WG and GP models to the percentage of Phosphorus concentration in leaves data.

Table 6
The failure times of 20 mechanical components.

0.067	0.068	0.076	0.081	0.084	0.085	0.085	0.086	0.089	0.098
0.098	0.114	0.114	0.115	0.121	0.125	0.131	0.149	0.160	0.485

As a second application, we consider the data consisting of the failure times of 20 mechanical components given in Murthy et al. (2004) and listed in Table 6. Obviously, due to the genesis of the EW family, the failure times are ideally modeled by this distribution. Thus, the use of the EWPS class for fitting these data is justified.

Table 7
Descriptive statistics.

Min.	Q_1	Q_2	Mean	Q_3	Max.	Var.
0.0670	0.0848	0.0980	0.1216	0.1220	0.4850	0.0080

Table 8
MLEs of the parameters with corresponding SE's (given in parentheses) and maximized log-likelihoods of the WG, GP and CP models for the second data set. The statistics AIC, BIC and CAIC are also displayed.

Model	$\widehat{\theta}$	$\widehat{\alpha}$	$\widehat{\gamma}$	AIC	BIC	CAIC	K–S	$-2\ell(\widehat{\Theta})$
WG	0.9999	8.1443	5.0876	−66.4	−63.4	−64.9	0.1810	−72.4
	(0.0001)	(0.0137)	(0.8002)					
	$\widehat{\theta}$	$\widehat{\alpha}$	$\widehat{\beta}$					
GP	5.4566	0.9909	6.5683	−41.9	−38.9	−40.4	0.3312	−47.9
	(2.4140)	(0.5504)	(2.2144)					
	$\widehat{\theta}$	$\widehat{\alpha}$	$\widehat{b}$					
CP	6.2426	25.3554	2.3796	−54.7	−51.7	−53.2	0.2214	−60.7
	(2.2755)	(15.8907)	(0.3380)					

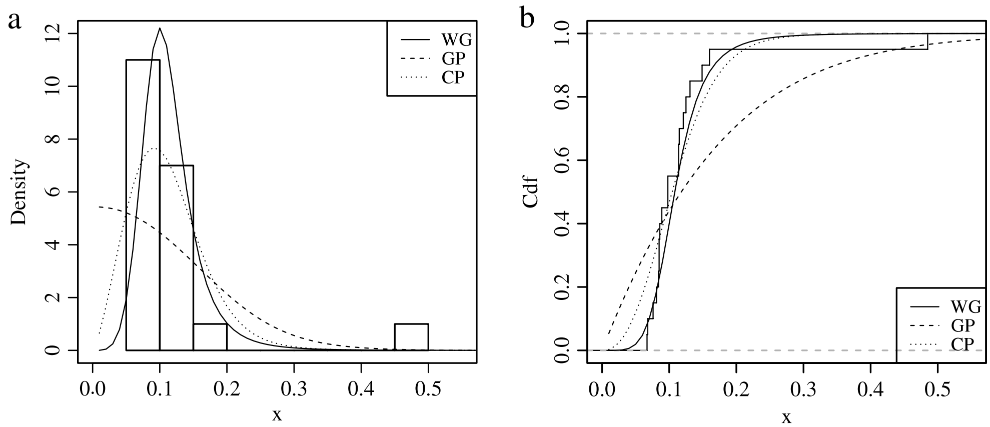


Fig. 8. Estimated (a) pdf and (b) cdf for the WG, GP and CP models to the failure times.

Table 7 display some descriptive statistics. The MLEs of the parameters (standard errors between parentheses), the Kolmogorov–Smirnov statistic, $-2\ell(\widehat{\Theta})$ and the values of the AIC, BIC and CAIC statistics are listed in Table 8. The values of these statistics indicate that the WG model yields a better fit to these data than the GP and CP models.

Plots of the estimated pdf and cdf of the fitted WG, GP and CP models to these data are displayed in Fig. 8. They indicate that the WG distribution is superior to the other distributions in terms of model fitting. From these figures, we conclude that this distribution provides a better fit to these data than the GP and CP models.

8. Concluding remarks

We define a new class of lifetime distributions called the extended Weibull power series (EWPS) class, which generalizes the Weibull power series class of distributions (Morais and Barreto-Souza, 2011). Further, the new class extends the exponential power series distributions (Chahkandi and Ganjali, 2009). We provide a mathematical treatment of the new class including expansions for the density function, moments, generating function, incomplete moments and reliability. Further, explicit expressions for the order statistics and Shannon entropy are derived. The EWPS density function can be expressed as a mixture of extended Weibull (EW) density functions. This mixture representation is important to derive several properties of the new class. Maximum likelihood inference is implemented straightforwardly for estimating the model parameters. We obtain the observed information matrix. Maximum entropy identification was discussed and some special models are explored. We fit some EWPS distributions to two real data sets to show the usefulness of the proposed class. In conclusion: we define a general approach for generating new lifetime distributions, at least 68 distributions, some of them known and the great majority new ones. Further, we motivate the use of the new class in four different ways. We think these two facts combined may attract more complex applications in the literature of lifetime distributions. Finally, the formulas derived are manageable by using modern computer resources with analytic and numerical capabilities.

Acknowledgments

The authors gratefully acknowledge financial support from CAPES and CNPq. The authors are also grateful to two referees and an associate editor for helpful comments and suggestions.

Appendix

The elements of the $p \times p$ information matrix $J_n(\Theta)$ are

$$J_{\theta\theta} = -\frac{n}{\theta^2} - n\left[\frac{C''(\theta)}{C(\theta)} - \left(\frac{C'(\theta)}{C(\theta)}\right)^2\right] + \theta\sum_{i=1}^{n}\left(\frac{z_{2i}}{z_{1i}}\right)^2 H(x_i;\,\xi)\mathrm{e}^{-2\alpha H(x_i;\,\xi)} - \theta\sum_{i=1}^{n}\frac{z_{3i}}{z_{1i}}H(x_i;\,\boldsymbol{\xi})\mathrm{e}^{-2\alpha H(x_i;\,\boldsymbol{\xi})}$$

$$J_{\alpha\alpha} = -\frac{n}{\alpha^2} + \theta\sum_{i=1}^{n}\frac{z_{2i}}{z_{1i}}H^2(x_i;\,\boldsymbol{\xi})\mathrm{e}^{-\alpha H(x_i;\,\xi)} + \theta^2\sum_{i=1}^{n}\frac{(z_{3i}-z_{2i}^2)}{z_{1i}}H^2(x_i;\,\xi)\mathrm{e}^{-2\alpha H(x_i;\,\xi)}$$

$$J_{\alpha\theta} = \theta\sum_{i=1}^{n}\left[\left(\frac{z_{2i}}{z_{1i}}\right)^2 - \frac{z_{3i}}{z_{1i}}\right]H^2(x_i;\,\xi)\mathrm{e}^{-2\alpha H(x_i;\,\xi)} - \sum_{i=1}^{n}\frac{z_{2i}}{z_{1i}}H^2(x_i;\,\xi)\mathrm{e}^{-\alpha H(x_i;\,\xi)}$$

$$\begin{aligned} J_{\alpha\xi_k} \;=\; & -\sum_{i=1}^{n}\frac{\partial H(x_i;\,\boldsymbol{\xi})}{\partial \xi_k} - \theta\sum_{i=1}^{n}\frac{z_{2i}}{z_{1i}}\frac{\partial H(x_i;\,\xi)}{\partial \xi_k}\mathrm{e}^{-\alpha H(x_i;\,\boldsymbol{\xi})}\left[1-\alpha H(x_i;\,\boldsymbol{\xi})\right] \\ & + \alpha\theta^2\sum_{i=1}^{n}\left[\frac{z_{3i}}{z_{1i}} - \left(\frac{z_{2i}}{z_{1i}}\right)^2\right]\frac{\partial H(x_i;\,\xi)}{\partial \boldsymbol{\xi}_k}H(x_i;\,\xi)\mathrm{e}^{-2\alpha H(x_i;\,\boldsymbol{\xi})} \end{aligned}$$

$$J_{\theta\xi_k} = \theta\alpha\sum_{i=1}^{n}\left[\left(\frac{z_{2i}}{z_{1i}}\right)^2 - \frac{z_{3i}}{z_{1i}}\right]\frac{\partial H(x_i;\,\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_k}\mathrm{e}^{-2\alpha H(x_i;\,\xi)} - \alpha\sum_{i=1}^{n}\frac{z_{2i}}{z_{1i}}\frac{\partial H(x_i;\,\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_k}\mathrm{e}^{-\alpha H(x_i;\,\boldsymbol{\xi})}$$

$$\begin{aligned} J_{\xi_k\xi_l} \;=\; & -\alpha\sum_{i=1}^{n}\frac{\partial^2 H(x_i;\,\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_k\partial \boldsymbol{\xi}_l} - \sum_{i=1}^{n}\frac{1}{H(x_i;\,\boldsymbol{\xi})^2}\frac{\partial H(x_i;\,\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_k}\frac{\partial H(x_i;\,\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_l} + \sum_{i=1}^{n}\frac{1}{H(x_i;\,\boldsymbol{\xi})}\frac{\partial^2 H(x_i;\,\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_k\partial \boldsymbol{\xi}_l} \\ & + (\alpha\theta)^2\sum_{i=1}^{n}\left[\left(\frac{z_{2i}}{z_{1i}}\right)^2 + \frac{z_{3i}}{z_{1i}}\right]\frac{\partial H(x_i;\,\boldsymbol{\xi})}{\partial \xi_k}\frac{\partial H(x_i;\,\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_l}\mathrm{e}^{-2\alpha H(x_i;\,\boldsymbol{\xi})} \\ & - \alpha\theta\sum_{i=1}^{n}\frac{z_{2i}}{z_{1i}}\frac{\partial^2 H(x_i;\,\xi)}{\partial \boldsymbol{\xi}_k\partial \boldsymbol{\xi}_l}\mathrm{e}^{-\alpha H(x_i;\,\boldsymbol{\xi})} + \alpha^2\theta\sum_{i=1}^{n}\frac{z_{2i}}{z_{1i}}\frac{\partial H(x_i;\,\xi)}{\partial \boldsymbol{\xi}_k}\frac{\partial H(x_i;\,\boldsymbol{\xi})}{\partial \boldsymbol{\xi}_l}\mathrm{e}^{-\alpha H(x_i;\boldsymbol{\xi})} \end{aligned}$$

where $z_{1i} = C'(\theta\mathrm{e}^{-\alpha H(x_i;\,\boldsymbol{\xi})})$, $z_{2i} = C''(\theta\mathrm{e}^{-\alpha H(x_i;\,\boldsymbol{\xi})})$ and $z_{3i} = C'''(\theta\mathrm{e}^{-\alpha H(x_i;\,\boldsymbol{\xi})})$, for $i = 1, \ldots, n$.

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